

**BETACOOOL Code Development  
and  
Beam Dynamics Simulations**

**Final report**

to the ANNEX No. 2

of the Memorandum of Understanding for FAIR

MoU-GSI-ACC-2005-01

between

Gesellschaft für Schwerionenforschung mbH (GSI), Darmstadt, Germany

and

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## Abstract

This report summarizes results of the software development during the Contract realization from March 1, 2007 to November 1, 2007. As it was agreed JINR made available to GSI a manual of the special program version and the results of beam simulations.

General attention at this stage of the work was devoted to development of following tasks:

- version management (CVS)
- user guide for work without interface and for WINDOWS interface,
- description the model for RF bucket simulation,
- description of the code modifications and results of benchmarking,
- propositions for further development.

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## 1. Version of BETACOOOL code for CVS server

There is a possibility to have access to Betacool sources in a frame of Concurrent Versions System (CVS).

Description of CVS using you can find on our web-site  
<http://lepta.jinr.ru/betacool/sources/sources.htm>.

You can work with CVS as under Linux as under Windows too.

At the present time CVS-server contains two version of Betacool sources: GSI 2.0 – version of 2006 and GSI 3.0 - version of 2007 which consist of:

- DAT – input and output files
- SRC – source files for C++ Compilers

### 1.1. CVS under Linux

Betacool sources downloading

1. Create folder where you plan to keep of Betacool program
2. Open terminal window from this folder
3. Type next command in it:  
`cvs -d :ext:gsi@lepta.jinr.ru:/srv/cvsroot checkout ??? betacool,`  
where ??? – name of version (2.0 or 3.0).
4. Insert password Germany
5. Betacool sources will be download at your computer.

Structure of the project:

dat	- folder
src	- folder
makefile	- file
makefile.config	- file

Next you can compile sources and start Betacool.

1. Go to folder src
2. Open terminal window in it and type command `make`
3. If you will get error it needs to do some changes:  
Once needs to create folders lib and lib/obj, then you have to get next project structure:

dat	- folder
lib	- folder
/obj	- folder
src	- folder
makefile	- file
makefile.config	- file

Then repeat steps 1 and 2.

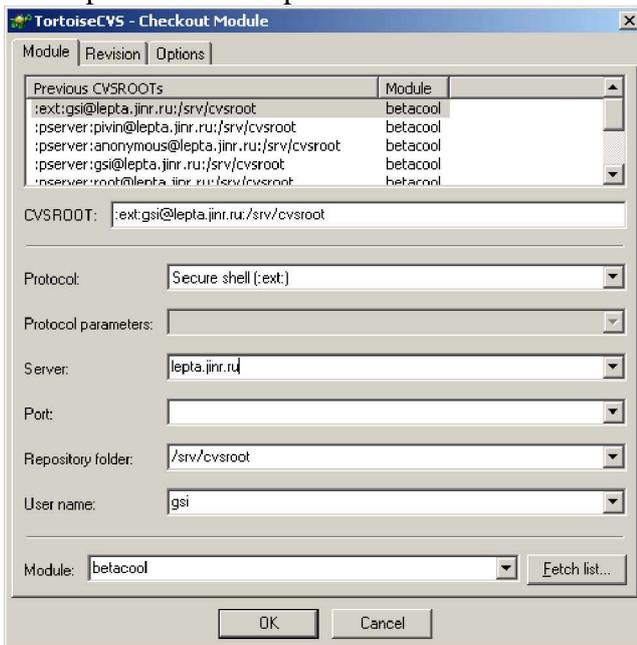
4. If you will not get error go to folder dat and run `./betacool nesr.bld /d` in terminal window, where nesr.bld – NESR ring, /d – RMS dynamics algorithm.

## 1.2. CVS under Window

1. Go to <http://lepta.jinr.ru/betacool/sources/sources.htm>
2. Download and install CVS-client for Windows TortoiseCVS1.8.31
3. Create folder where you plan to keep of Betacool program
4. Use right mouse button and choose CVS Checkout command.



### 5. Set parameters in opened window



6. After you push "Ok", it will be necessary to insert password "Germany".
7. Betacool sources will be download at your computer.

## 2. Core-Tail model of intrabeam scattering kick

### 2.1. Calculation of heating rates for Bi-Gaussian model

Model Beam algorithm can simulate the beam dynamics with arbitrary distribution of model particles. The most number of physical effects in the BETACOOOL program make kicks on the ion beam in the single particle approximation. It means that particle distribution does not influence on the kick value from each effects.

Another situation exists for IBS effect when heating rates are calculated as the collective effect. Standard model of IBS is assumed that the particles have the Gaussian distribution. In the case of Model Beam algorithm the particles can have the arbitrary distribution. To resolve this problem the special Bi-Gaussian approximation was elaborated. The distribution function can be presented as summary of two Gaussian functions for core and tail particles:

$$f(x_i) = f^{tail}(x_i) + f^{core}(x_i) = a_i^{tail} \exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_i^{tail}}\right)^2\right] + a_i^{core} \exp\left[-\frac{1}{2}\left(\frac{x_i}{\sigma_i^{core}}\right)^2\right], \quad (2.1)$$

where  $a^{tail}$ ,  $a^{core}$ ,  $\sigma^{tail}$ ,  $\sigma^{core}$  – amplitudes and widths of beam profile,  $i$  – corresponds to degrees of freedom: 1 – horizontal, 2 – vertical, 3 – longitudinal. Powell method [9] is used in the code to find of Gaussian function parameters. This method minimizes the deviation between the model beam profile and Bi-Gaussian distribution:

$$\sum_{i=1}^N [y_i - f(x_i)]^2 \rightarrow \text{Minimum}, \quad (2.2)$$

where  $(x_i, y_i)$  – points of the beam profile distribution,  $N$  is the total particle number. RMS parameters of ion beam  $E_i^{rms}$  (emittances and momentum spread) are calculated in the suggestion that distribution of model particles has Gaussian shape. Beam parameters and particle number for Bi-Gaussian distributions are:

$$E_i^{tail} = E_i^{rms} (\sigma_i^{tail})^2, \quad E_i^{core} = E_i^{rms} (\sigma_i^{core})^2, \quad N_i^{core} = N \times \left[1 + \frac{\sigma_i^{tail}}{\sigma_i^{core}} \cdot \frac{a_i^{tail}}{a_i^{core}}\right]^{-1}, \quad N_i^{tail} = 1 - N_i^{core}. \quad (2.3)$$

Heating growth rates  $\tau_i^{tail}(E_i^{tail}, N)$  and  $\tau_i^{core}(E_i^{core}, N)$  of Bi-Gaussian distributions are calculated with standard procedure as described in Physics Guide on the base of choosing IBS model for Gaussian distribution. Model particles in the tail get the kick with standard procedure as described in section {5.1} using of tail heating growth rates.

$$\Delta\theta_{1,2}^{tail} = \sqrt{2 \frac{\varepsilon_{1,2}^{tail}}{\beta_{1,2}} \frac{T_{rev}}{\tau_{1,2}^{tail}} N_{turn}} \times \xi, \quad \Delta\theta_3^{tail} = \sqrt{\left(\frac{\Delta p^{tail}}{p}\right)^2 \frac{T_{rev}}{\tau_3^{tail}} N_{turn}} \times \xi, \quad (2.4)$$

where  $\varepsilon_{1,2}$  – horizontal and vertical emittances,  $\beta_{1,2}$  – horizontal and vertical beta-functions,  $(\Delta p/p)$  – momentum spread,  $T_{rev}$  – revolution period,  $N_{turn}$  – number of turns,  $\xi$  – the random value with Gaussian distribution at unit dispersion. Particles in the core get the summary kick from core and tail heating rates:

$$\begin{aligned} \frac{1}{\tau_i^{sum}} &= \frac{1}{\tau_i^{core}} \left(\frac{N_i^{core}}{N}\right)^2 + \frac{1}{\tau_i^{tail}} \left(\frac{N_i^{tail}}{N}\right)^2 \frac{E_i^{tail}}{E_i^{core}}. \\ \Delta\theta_{1,2}^{core} &= \sqrt{2 \frac{\varepsilon_{1,2}^{core}}{\beta_{1,2}} \frac{T_{rev}}{\tau_{1,2}^{sum}} N_{turn}} \times \xi \\ \Delta\theta_3^{core} &= \sqrt{\left(\frac{\Delta p^{core}}{p}\right)^2 \frac{T_{rev}}{\tau_3^{sum}} N_{turn}} \times \xi \end{aligned} \quad (2.5)$$

## 2.2. Parameters of Core-Tail model

Betacool code includes two variants of Core-Tail model in the present version: Bi-Gaussian and Full Width on Half Maximum (FWHM). For FWHM model the tail parameters are calculated from the RMS definition of emittances. User can choose the necessary variant of model in the DropBox IBS model of Bi-Gaussian TabSheet (Fig.2.1). Core definition [sigma] defines the range (in sigma units) of the distribution in which particle are indicated as in core. Invariants in core [1-3] – number of invariants which are used for the definition of particles in core (it can have value 1, 2 and 3). Even user choose Bi-Gaussian model of IBS however for the core definition one can choose the FWHM definition with parameter Core from FWHM. Note that for the coasting beam the longitudinal invariant always use FWHM definition for core particles.

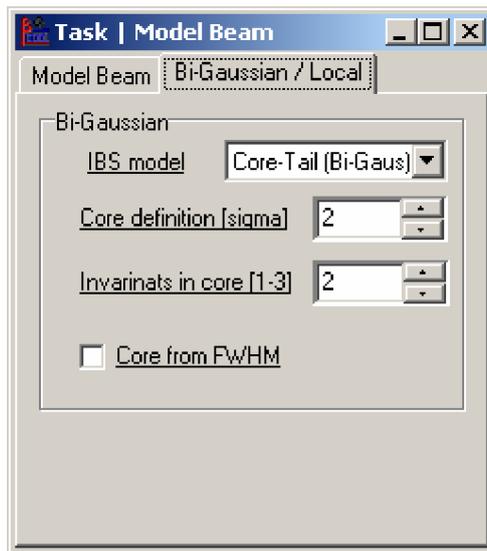


Fig.2.1. Form for parameters of Bi-Gaussian model of intrabeam scattering kick.

During simulation process user can check how many particles are indicated as in core. Parameter Ncore[%] in the console of the Betacool program shows this value (Fig.2.2) which can be compared with the sorted and unsorted invariants (Fig.2.3). If the large disagreement exists between definitions of particles in the core than one can adjust parameters of the Core-Tail model on the Bi-Gaussian Form (Fig.2.1).

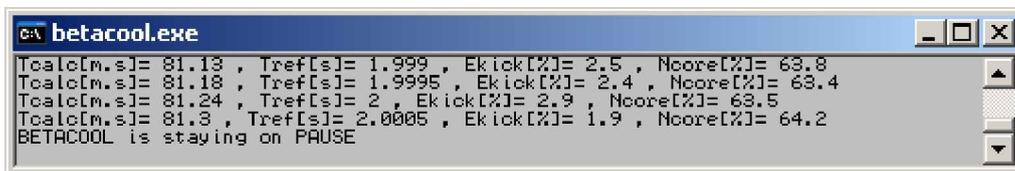


Fig.2.2. Console window of Betacool program

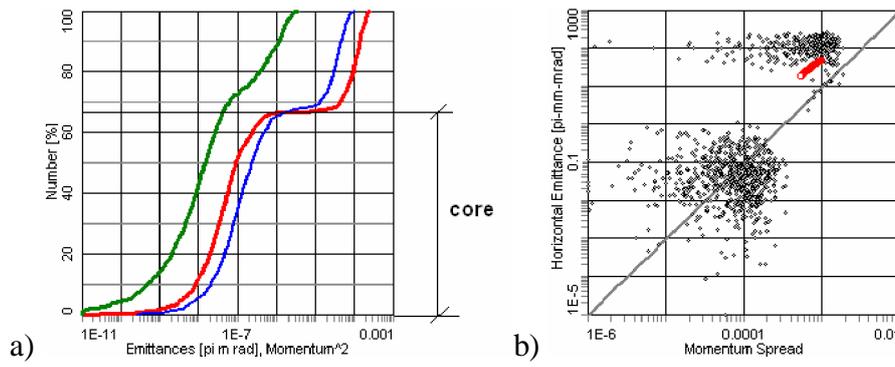


Fig.2.3. Example of invariant distributions for SIS18 parameters (see Section 4.1).  
 a) sorted invariants: green – longitudinal, red – transverse, blue – vertical;  
 b) black dots – unsorted invariants, gray line – equilibrium of transverse and longitudinal temperatures, red dots – dependence of transverse emittance on momentum spread.

### 3. Barrier RF Bucket simulation

The barrier-bucket system application is one of the effective ways to achieve a high luminosity in experiment with internal target simultaneously with small phase space volume of the beam. One of the general effects leading to dilution of the beam quality at interaction with a target is mean energy loss of the particles. This effect sufficiently limits an available target thickness if a maximum cooling power is fixed. If the mean energy losses are compensated, at the same cooling power one can sufficiently increase the target thickness and, correspondingly, the luminosity.

The mean energy loss can be compensated by usual RF system at relatively small voltage amplitude; however this leads to sufficient increase of IBS growth rates. Even at long length of the bunch the particle density in its central part increases significantly in comparison with a coasting beam.

At a barrier-bucket application the particle density inside the bucket is almost uniform. Therefore the IBS growth rates increase by a factor equal to ratio of the ring circumference to the bucket length only. However in difference with a coasting beam the barrier-bucket system compensates the mean energy loss in the target and leads also to increase of an electron cooling efficiency. The last fact is connected with peculiarity of the particle synchrotron motion at barrier-bucket application: when the particle is reflected by the barrier its momentum deviation crosses zero and this corresponds to significant increase of the friction force acting on the particle inside the electron beam.

In the frame of RMS dynamics algorithm a shape of distribution function can not be investigated and the algorithm required only slight modifications, if an equation of the phase trajectory is known. Evolution of the distribution function shape and particle loss from a bucket can be investigated in Model Beam algorithm. For this a solution of the motion equation inside and outside the bucket is necessary. General behaviors of the motion are determined by integrated pulse strength, and essential physics is independent on the exact shape of the barrier RF wave. Simplest analytical solution for the phase trajectory can be obtained at square wave barrier bucket; therefore this model can be implemented into the program at the first step.

#### 3.1. Synchrotron motion in a square wave barrier bucket

The RF voltage time dependence and phase trajectories of ions are sketched in the Fig. 3.1. When the ion passes through the cavity gap at voltage  $\pm V_0$  it gains (losses) an equal amount of energy  $ZeV_0$ , i.e.

$$\frac{d(\Delta E)}{dt} = \pm \frac{ZeV_0}{T_0}, \quad (3.1)$$

where  $\Delta E$  is the energy deviation from synchronous one,  $T_0$  – revolution period. The ion trajectory in the longitudinal phase space  $(t-t_0, \Delta E)$  inside the bucket can be written in the following form:

$$(\Delta E)^2 = \begin{cases} A_E^2 & \text{if } |t-t_0| \leq T_2/2 \\ A_E^2 - \left( |t-t_0| - \frac{T_2}{2} \right) \frac{2\beta^2 E_0 ZeV_0}{T_0 |\eta|} & \text{if } T_2/2 \leq |t-t_0| \leq (T_2/2) + T_1 \end{cases} \quad (3.2)$$

where  $A_E$  is the maximum energy deviation from synchronous energy  $E_0$ ,  $\beta c$  is the ion velocity,  $\eta$  is the ring off-momentum factor. The phase space trajectory is composed of a straight line in the RF gap region and a parabola in the square RF wave region.

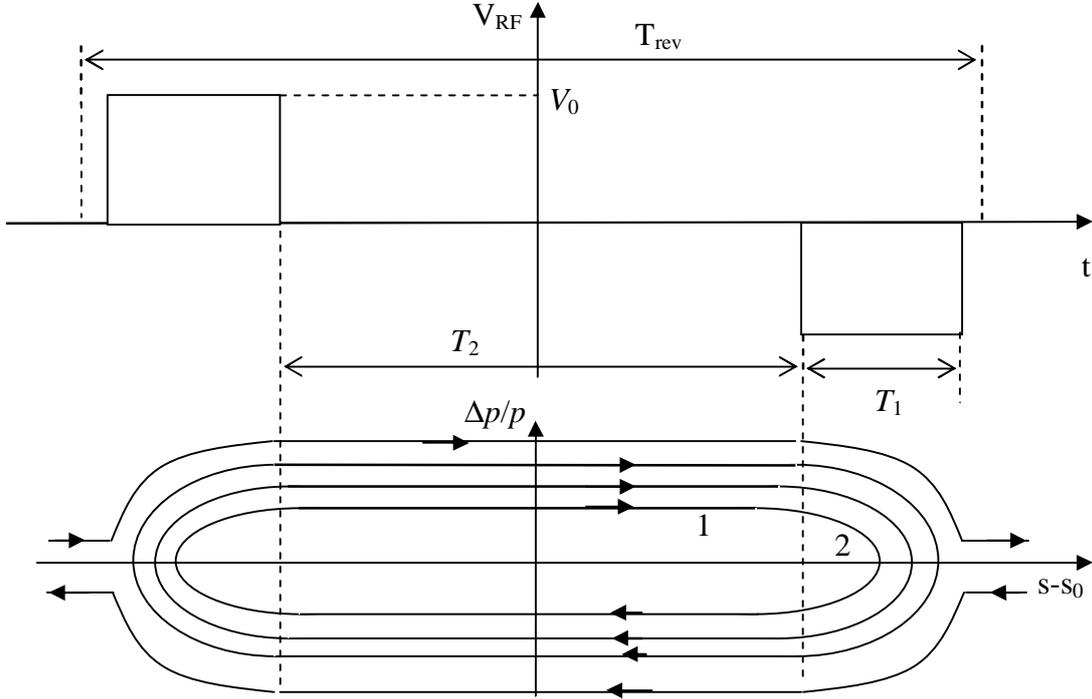


Fig. 3.1. RF voltage and particle trajectories in the longitudinal phase space at square wave barrier-bucket.  $V_0$  is the voltage height,  $T_1$  is the pulse width,  $T_2$  is the gap duration.

The maximum energy deviation or the barrier height that a barrier RF wave can provide is

$$\Delta E_b = \left( \frac{ZeV_0T_1}{T_0} \frac{2\beta^2 E_0}{|\eta|} \right)^{1/2}. \quad (3.3)$$

The period of synchrotron oscillations is

$$T_s = 2 \frac{T_2}{|\eta|} \left( \frac{\beta^2 E_0}{|A_E|} \right) + 4 \frac{|A_E|}{ZeV_0} T_0. \quad (3.4)$$

The relation between rms energy spread  $\sigma_E$  and maximum energy deviation of “rms ion” can be found from the following equation:

$$\frac{1}{T_s} \int_0^{T_s} (\Delta E)^2 dt = \sigma_E^2. \quad (3.5)$$

When the maximum energy deviation of the “rms” particle is known one can estimate the bucket length as

$$S_b = \left( \frac{T_2}{T_0} + \frac{A_{E,rms}^2}{\beta^2 E_0 ZeV_0} \right) C, \quad (3.6)$$

where  $C$  is the ring circumference.

To implement the Barrier Bucket simulation into BETACOOL code the equation of the synchrotron motion has to be rewritten in the phase space ( $s-s_0$ ,  $\delta = \frac{\Delta p}{p}$ ). The equation is

$$\begin{cases} \frac{d(s-s_0)}{dt} = |\eta| \beta c \delta_A \\ \frac{d\delta}{dt} = 0 \end{cases} \quad (3.7)$$

at the part of the trajectory 1 (Fig. 1), and

$$\begin{cases} \frac{d(s-s_0)}{dt} = |\eta| \beta c \delta \\ \frac{d\delta}{dt} = -\frac{ZeV_0}{Cp_0} \end{cases} \quad (3.8)$$

at the part 2. Here  $\delta_A$  is the amplitude of the momentum deviation,  $C$  is the ring circumference,  $p_0$  – synchronous momentum.

The time of flight of the first part of the trajectory can be found from the first equation of (7) using the condition  $(s-s_0)(t_1) = \frac{\beta c T_2}{2}$ :

$$t_1 = \frac{T_2}{2|\eta|\delta_A}. \quad (3.9)$$

Solution of the second equation of the system (8)

$$\delta = \delta_A - \frac{ZeV_0}{Cp_0} t \quad (3.10)$$

permits to calculate time of flight the trajectory part 2 from the condition  $\delta(t_2) = 0$

$$t_2 = \frac{Cp_0\delta_A}{ZeV_0}. \quad (3.11)$$

The period of synchrotron oscillations in this notation is

$$T_s = 4(t_1 + t_2) = \frac{2T_2}{|\eta|\delta_A} + \frac{4Cp_0\delta_A}{ZeV_0}, \quad (3.12)$$

which coincides with (4). From the equations (3.9), (3.11) the  $\sigma_\delta^2$  can be expressed as

$$\sigma_{\delta}^2 = \frac{t_1 \delta_A^2 + \int_0^{t_2} \delta^2 d\tau}{t_1 + t_2}.$$

where  $\delta$  is given by (3.10). That leads to

$$\frac{(2/3)\eta|Cp_0\delta_A^4 + ZeV_0T_2\delta_A^2}{2|\eta|Cp_0\delta_A^2 + ZeV_0T_2} = \sigma_{\delta}^2.$$

Introducing

$$\xi^2 = \frac{ZeV_0T_2}{2|\eta|Cp_0} \quad (3.13)$$

one can obtain

$$\delta_A^2 = \frac{\sqrt{9(\xi^2 - \sigma_{\delta}^2)^2 + 12\xi^2\sigma_{\delta}^2} - 3(\xi^2 - \sigma_{\delta}^2)}{2}. \quad (3.14)$$

If this value is larger than the barrier height the particle oscillates outside the bucket and formula (3.14) is not correct. The maximum momentum deviation  $\delta_{A,\max}$  corresponding to the barrier height can be calculated using solution of the motion equation in the 2 part of the trajectory:

$$s - s_0 = |\eta|\beta c \delta_A t - \frac{|\eta|\beta c ZeV_0}{2Cp_0} t^2. \quad (3.15)$$

The maximum momentum deviation satisfies to the equation  $(s - s_0)(t_2) = \beta c T_1$ , that gives

$$\delta_{A,\max}^2 = \frac{2T_1}{T_0} \frac{ZeV_0}{\beta cp_0 |\eta|}, \quad (3.16)$$

which coincides with (3) after required substitutions. The distance which particles crosses inside the RF wave is

$$(s - s_0)_{in} = |\eta|\beta c \delta_A t_2 - \frac{|\eta|\beta c ZeV_0}{2Cp_0} t_2^2 = \frac{|\eta|\beta c Cp_0 \delta_A^2}{2ZeV_0},$$

and total trajectory length is

$$S_b = T_2 \beta c + \frac{|\eta|\beta c Cp_0 \delta_A^2}{ZeV_0}. \quad (3.17)$$

For a particle outside the bucket ( $\delta_A > \delta_{A,\max}$ ) the time of flight through the decelerating RF pulse can be found from quadratic equation (3.15) with  $(s - s_0)(t) = \beta c T_1$ :

$$t_d = t_2 \left( 1 - \sqrt{1 - \frac{\delta_{A,\max}^2}{\delta_A^2}} \right), \quad (3.18)$$

where  $t_2$  is determined by (3.11). The momentum deviation after crossing the decelerating wave is equal to

$$\delta_M = \delta_A \sqrt{1 - \frac{\delta_{A,\max}^2}{\delta_A^2}}. \quad (3.19)$$

When this particle crosses the accelerating wave the solution of the motion equation is

$$s - s_0 = |\eta| \beta c \delta_M t + \frac{|\eta| \beta c Z e V_0}{2 C p_0} t^2,$$

and the time of flight is

$$t_{acc} = t_2 \frac{\delta_M}{\delta_A} \left( \sqrt{1 + \frac{\delta_{A,\max}^2}{\delta_M^2}} - 1 \right) = t_d. \quad (3.20)$$

Momentum deviation after crossing the second wave is

$$\delta = \delta_M \sqrt{1 + \frac{\delta_{A,\max}^2}{\delta_M^2}} = \delta_A. \quad (3.21)$$

The time of flight through the second gap between RF waves is

$$t_g = \frac{T_0 - T_2 - 2T_1}{|\eta| \delta_M}$$

and the period of synchrotron oscillations is

$$T_{s,out} = 2t_1 + t_d + t_g + t_{acc} = \frac{T_2}{|\eta| \delta_A} + \frac{2Cp_0 \delta_A}{ZeV_0} \left( 1 - \sqrt{1 - \frac{\delta_{A,\max}^2}{\delta_A^2}} \right) + \frac{T_0 - T_2 - 2T_1}{|\eta| \sqrt{\delta_A^2 - \delta_{A,\max}^2}}. \quad (3.22)$$

For the particle at given longitudinal co-ordinates ( $s-s_0$ ,  $\delta$ ) the amplitude of oscillations can be calculated in accordance with the phase trajectory equation. The result is obvious, when ( $s-s_0$ ) lies in one of the gaps between RF waves. Inside the wave it is equal to

$$\delta_A^2 = \delta^2 + \frac{2ZeV_0}{|\eta| \beta c C p_0} (s - s_0), \quad (3.23)$$

where ( $s-s_0$ ) is measured from the beginning of the wave. Knowing the amplitude the particle co-ordinates at random phase of the oscillation can be generated by generation of random number uniformly distributed between 0 and  $T_s$  (or  $T_{s,out}$ , when  $\delta_A > \delta_{A,\max}$ ) and calculate momentum deviation and longitudinal co-ordinate in accordance with equation of the phase space trajectory.

## 3.2. Modifications of RMS dynamics algorithm

### 3.2.1. Internal target and IBS

Simulations of intrabeam scattering and interaction with internal target require only slight modifications. At interaction with a target the mean energy loss has to be excluded and characteristic growth time for momentum spread should be calculated using expression for the energy loss fluctuations. However in presence of dispersion in the target position the emittance deviation has to be calculated taking into account mean energy loss:

$$\begin{cases} \Delta\varepsilon_{h,v} = \frac{\beta_{h,v}\theta_{rms}^2}{2} + \left( \frac{1+\alpha_{h,v}^2}{\beta_{h,v}} D_{h,v}^2 + 2\alpha_{h,v} D_{h,v} D'_{h,v} + \beta_{h,v} D'_{h,v}{}^2 \right) \times \left( \frac{\Delta p}{p} \right)_{mean}^2 \\ \Delta\varepsilon_l = \left( \frac{\Delta p}{p} \right)_{str}^2 \end{cases} \quad (3.24)$$

Here  $\varepsilon_{h,v}$  are the transverse rms emittances. Longitudinal emittance of the ion beam is determined in the program as a mean square of the particle momentum deviation:

$$\varepsilon_l = \langle \delta^2 \rangle. \quad (3.25)$$

$\theta_{rms}^2$ ,  $\left( \frac{\Delta p}{p} \right)_{mean}$  and  $\left( \frac{\Delta p}{p} \right)_{str}$  are the rms scattering parameters – scattering angle and momentum deviation due to mean energy loss and energy loss fluctuations.  $\beta_{h,v}$ ,  $\alpha_{h,v}$  are horizontal and vertical beta and alpha functions,  $D_{h,v}$ ,  $D'_{h,v}$  are dispersions and derivatives of the dispersion in the target position.

To realize this algorithm the procedure `vectorU xTarget::Rates(xTime&time, xBeam&beam, xRing&ring)` was modified.

The IBS process in the first approximation can be simulated using formulae for coasting beam at substitution of the bucket length instead the ring circumference. The bucket length can be calculated in the following steps: for given longitudinal emittance to calculate amplitude of the rms particle oscillations using formula (3.14), for this amplitude to calculate the bucket length in accordance with (3.17).

### 3.2.2. Electron cooling time calculation

The characteristic cooling time in the frame of rms dynamics model is determined as a change of the rms particle invariant of the motion after crossing the cooling section. It is calculated in accordance with:

$$\frac{1}{\tau_{cool}} = \frac{1}{I T_{rev}} \langle \delta I \rangle, \quad (3.26)$$

where  $T_{rev}$  is the particle revolution period in the storage ring, the brackets mean averaging over the phases of betatron and synchrotron oscillations:

$$\langle \delta \vec{I} \rangle = \frac{1}{8\pi^3} \int_0^{2\pi} \int \int \delta \vec{I}(\vec{I}, \varphi_x, \varphi_y, \varphi_s) d\varphi_x d\varphi_y d\varphi_s, \quad (3.27)$$

where  $I_{x,y,l}$  are the corresponding invariants of the particle motion.

The Courant – Snider invariants of the “rms particle” relate to the rms beam emittance as

$$I_{x,y} = 2\varepsilon_{x,y}. \quad (3.28)$$

In the case of **coasting beam** the cooling time is calculated by averaging over two values of the momentum deviation:

$$\delta = \pm \sqrt{\varepsilon_l}. \quad (3.29)$$

In the linear approximation, for a **bunched beam** one can express the particle momentum deviation and its longitudinal co-ordinate (distance from the bunch center) as a function of the phase of synchrotron oscillations:

$$\delta = \sqrt{I_l} \cos \varphi_s, \quad s - s_0 = \beta_l \sqrt{I_l} \sin \varphi_s. \quad (3.30)$$

here the "synchrotron function" is determined as:

$$\beta_l = \frac{R|\eta|}{Q_s}, \quad (3.31)$$

where  $R$  is the mean ring radius,  $Q_s$  is the synchrotron tune:

$$Q_s = \frac{1}{\beta} \sqrt{\frac{ZehV|\eta|}{A2\pi m_p c^2 \gamma}}, \quad (3.32)$$

$h$  is the harmonic number,  $V$  is the RF voltage amplitude,  $\beta, \gamma$  are relativistic parameters,  $Z$  and  $A$  are the ion charge and atomic numbers,  $m_p$  is the proton mass,  $\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$ .

For the “rms particle” the invariant of the longitudinal motion is calculated as:  $I_l = 2\varepsilon_l$ .

In the case of **barrier bucket** the maximum momentum deviation of the “rms particle” can be calculated using (14). Averaging of the invariant deviation is simply to perform over  $t-t_0$  using the phase space trajectory equation (10), (15). The synchrotron period  $T_s$  in this algorithm is divided by a few equidistant intervals. In the time of  $T_i$  the particle co-ordinate and momentum deviation are calculated in accordance with the following formulae:

$$\text{if } 0 \leq T_i < t_1 \quad \begin{aligned} s - s_0 &= |\eta| \beta c \delta_A T_i \\ \delta &= \delta_A \end{aligned}$$

$$\begin{aligned}
& \text{if } t_1 \leq T_i < t_1 + 2t_2 & s - s_0 &= |\eta| \beta c \delta_A T_i - \frac{|\eta| \beta c Z e V_0}{2 C p_0} (T_i - t_1)^2 \\
& & \delta &= \delta_A - \frac{Z e V_0}{C p_0} (T_i - t_1) \\
& \text{if } t_1 + 2t_2 \leq T_i < 3t_1 + 2t_2 & s - s_0 &= |\eta| \beta c \delta_A (2t_1 + 2t_2 - T_i) \\
& & \delta &= -\delta_A \\
& \text{if } 3t_1 + 2t_2 \leq T_i < 3t_1 + 4t_2 & s - s_0 &= -\beta c \frac{T_2}{2} - |\eta| \beta c \delta_A (T_i - 3t_1 - 2t_2) + \frac{|\eta| \beta c Z e V_0}{2 C p_0} (T_i - 3t_1 - 2t_2)^2 \\
& & \delta &= \frac{Z e V_0}{C p_0} (T_i - 3t_1 - 2t_2) - \delta_A \\
& \text{if } 3t_1 + 4t_2 \leq T_i < T_s & s - s_0 &= |\eta| \beta c \delta_A (T_i - T_s) \\
& & \delta &= \delta_A
\end{aligned}$$

where  $t_1$  is given by formula (3.9),  $t_2$  – (3.11).

After crossing the cooling section one needs to recalculate new ion co-ordinates  $(s-s_0, \delta)$  into rms momentum deviation. The new amplitude of the particle oscillations  $\delta_A$  is calculated in accordance with (3.23) and corresponding rms momentum deviation is given by:

$$\langle \delta^2 \rangle = \frac{\delta_A^4 + 3\xi^2 \delta_A^2}{3(\delta_A^2 + \xi^2)}, \quad (3.33)$$

where  $\xi$  is given by (3.13).

This algorithm can give a correct number for the cooling rate only when the maximum momentum deviation of the “rms particle” is less than barrier height, determined by (3.16). In the frame of RMS dynamics the particle loss from the bucket can not be simulated correctly.

To realize this algorithm:

The procedure `vectorU xEcool::SingleParticle(xTime&t, xBeam& beam, xRing& ring)` was modified,

new variable ***bool bucket*** was introduced into ***xDistributor*** object with corresponding correction in constructor.

for variables  
***doubleU V\_B;***

*//Voltage amplitude*

***doubleU T1;***

*//RF duration in T0*

***doubleU T2;***

*//Gap duration in T0*

***doubleU BarHeight;***

*//Barrier height*

were introduced into ***xRing*** object with corresponding corrections in constructor and ***OnGet*** procedure.

Pointers for the ***xRing*** variables

```

doubleU* pBB_V_B;           //Voltage amplitude
doubleU* pBB_T1;           //RF duration in T0
doubleU* pBB_T2;           //Gap duration in T0
doubleU* pBB_BarHeight;    //Barrier height

```

the variables

```

doubleU Size_Bac;          //Bucket length
doubleU T_s;               //Synchrotron period
doubleU Amp2;              //Square of momentum amplitude
doubleU t_1;
doubleU t_2;
doubleU ksi2;

```

new procedures

```

void xBeam::CalcBucket()   //Calculation of the Bucket parameters
doubleU xBeam::Bdp2(doubleU s,doubleU dp) // Rms momentum via co-ordinates

```

were introduced into *xBeam* object, the constructor, *OnGet* and *OnSet* procedures were modified.

The input and output parameters for simulations were located in the input file in the following row

```

[row=26] Ring | Parameters | Barrier Bucket
30= Voltage amplitude, kV
0.1= RF duration in T0
0.7= Gap duration in T0
0.002621098342=Barrier height
1=RMS amplitude
1= Bucket length, m
1;Synchrotron period in sec

```

New tab sheet was added to the visual form *Ring / Parameters* (Fig. 3.2)

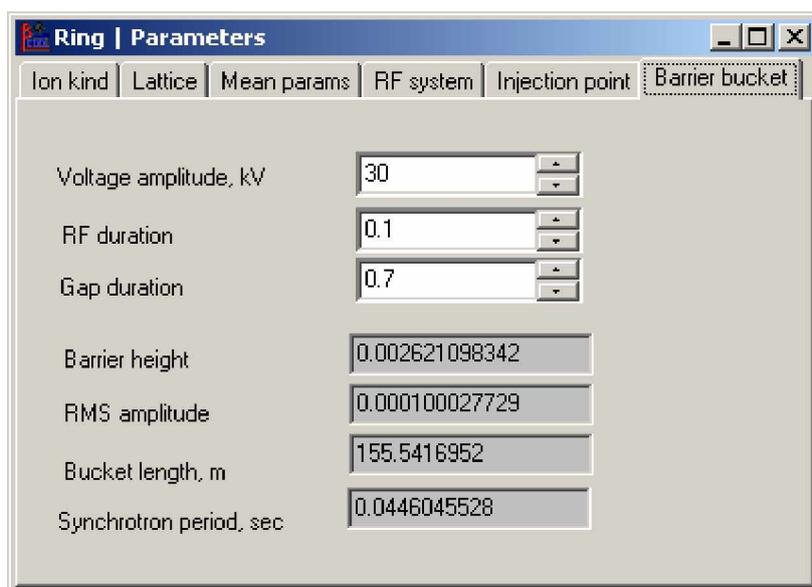


Fig. 3.2. Visual form for input and output the Barrier Bucket parameters.

### 3.3. Model beam algorithm

#### 3.3.1. Generation of the ion initial distribution in the given ring position

Generation of initial ion distribution in the longitudinal phase space, generation of new model particle at losses are based on procedures for generation of individual ion co-ordinates and matching of the ion with the ring lattice in the generation position.

The particle coordinates are generated under assumption of Gaussian shape of the distribution function in all degrees of freedom. Each particle co-ordinate is calculated in accordance with:

$$x_i = \sigma_{x_i} \sqrt{-2 \ln \xi_{1,i}} \sin(2\pi \xi_{2,i}), \quad (3.34)$$

$\xi_1$  and  $\xi_2$  are random values uniformly distributed in the interval from 0 to 1 generated with the standard random generator.

In the transverse planes one needs to match the particle coordinates with the lattice parameters of the ring. This procedure is realized by the following steps.

1. The particle “betatron” co-ordinates are distributed initially around a canonical ellipse at half-axis equal to the standard deviations for transverse co-ordinates and their derivatives (Fig. 3.3):

$$\sigma_x = \sqrt{\epsilon_x \beta_x}, \quad \sigma_{x'} = \sqrt{\epsilon_x \gamma_x}.$$

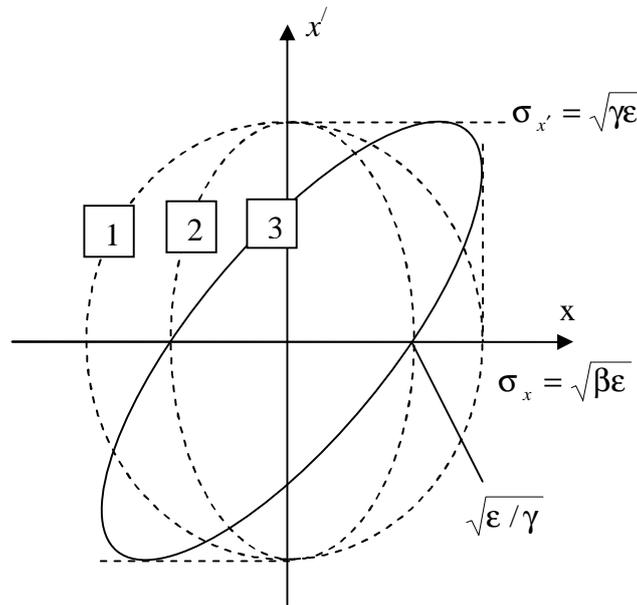


Fig. 3.3. Matching of the ion coordinate with the ring lattice in the transverse phase space.

2. Matching with the lattice parameters the derivatives of the particle transverse co-ordinates are not changed. The transverse “betatron” co-ordinates are changed:

$$x_\beta = \frac{x_{in}}{\sqrt{1 + \alpha_x^2}} - \frac{\alpha_x}{\gamma_x} x', \quad (3.35)$$

these transformations are shown schematically in the Fig. 2.

3. The matching with the dispersion and its derivative:

$$x = x_{\beta} + D_x \delta, \quad x' = x'_{\beta} + D'_x \delta. \quad (3.36)$$

For a **coasting beam** the particle momentum deviation is distributed in accordance with Gaussian law at standard deviation of  $\sigma_{\delta} = \sqrt{\epsilon_l}$ . The longitudinal co-ordinate is generated in accordance with:

$$s - s_0 = l_c \sqrt{3} \xi, \quad (3.37)$$

where  $l_c$  is the cell dimension for tracking algorithm,  $\xi$  is a random number uniformly distributed from -1 to 1. This algorithm is required for Molecular Dynamics simulations. In the Model Beam algorithm the longitudinal co-ordinate is not used. The cell length in the MD algorithm is usually sufficiently shorter than an optic element length and one can assume, that the lattice functions at the cell length are constants.

For a **bunched beam** in the longitudinal phase space the standard deviations of co-ordinate and momentum are calculated in accordance with the formulae:

$$\sigma_{\delta} = \sqrt{\epsilon_l}, \quad \sigma_s = \beta_l \sigma_{\delta}. \quad (3.38)$$

Generation of the ion co-ordinates in the case of **Barrier Bucket** can be realized in the following steps.

1. Initially the momentum deviation is generated in accordance with Gaussian law at standard deviation equal to  $\sqrt{\epsilon_l}$ .
2. For the given momentum deviation the maximum momentum deviation is calculated in accordance with the formula (14).
3. The phase of the synchrotron oscillation is generated to be uniformly distributed from 0 and  $2\pi$ . At given maximum deviation and phase the values of the momentum deviation and longitudinal co-ordinate are calculated using equation of the phase space trajectory.

If the absolute value of the particle momentum deviation is larger than the barrier height the particle circulates in the ring outside the bucket.

### 3.3.2. The betatron and synchrotron motion simulation

The Model Beam algorithm realizes Monte-Carlo simulation method of solution of Langevin equation based on an assumption, that the integration step over time is sufficiently longer than the decoherence period (a few thousands of revolutions). In this case the phase advance of betatron and synchrotron oscillations on the integration step is an arbitrary number. In the transverse phase space the betatron motion is simulated using linear transformation matrixes from one effect position to

another. The vector of ‘‘betatron’’ coordinates  $\begin{pmatrix} x_{\beta} \\ x'_{\beta} \end{pmatrix}$  ( $x_{\beta} = x - D\delta$ ,  $x'_{\beta} = x' - D'\delta$ ) is multiplied by the matrix with coefficients

$$R_{11} = \sqrt{\frac{\beta_2^x}{\beta_1^x}} (\cos \mu_x + \alpha_1^x \sin \mu_x), R_{12} = \sqrt{\beta_2^x \beta_1^x} \sin \mu_x$$

$$R_{21} = -\frac{(\alpha_2^x - \alpha_1^x) \cos \mu_x + (1 + \alpha_2^x \alpha_1^x) \sin \mu_x}{\sqrt{\beta_2^x \beta_1^x}}, R_{22} = \sqrt{\frac{\beta_1^x}{\beta_2^x}} (\cos \mu_x - \alpha_2^x \sin \mu_x),$$

where  $\beta_1, \alpha_1$  are the lattice functions in the first effect position,  $\beta_2, \alpha_2$  – in the second effect position,  $\mu_x$  is a random number uniformly distributed from 0 and  $2\pi$ . The same is for the vertical plane.

The synchrotron oscillations are simulated for a bunched beam only. The synchrotron motion is assumed to be linear (as well as the betatron one). The vector of longitudinal ion co-ordinates

$\begin{pmatrix} s - s_0 \\ \delta \end{pmatrix}$  is multiplied by the matrix

$$\begin{pmatrix} \cos \mu & \beta_l \sin \mu \\ -\frac{\sin \mu}{\beta_l} & \cos \mu \end{pmatrix}, \quad (3.39)$$

where  $\mu$  is a random number uniformly distributed from 0 and  $2\pi$ . The particle oscillations are provided between actions of effects, which change the particle momentum components in accordance with friction and diffusion components. The particle losses from the bunch are taking into account by introducing of two input parameters: the separatrix length  $L_{sep}$  and height  $A_l$  (longitudinal acceptance). The ion is lost when  $|s - s_0| > L_{sep} / 2$  or  $|\delta| > A_l$ . In this case instead the lost model particle the program generates new one using one of algorithm simulating the particle loss.

In the case of a **Barrier Bucket** the simulation of the synchrotron motion can be based on the phase space trajectory equation. At variation of the ion co-ordinates due to its synchrotron motion one needs to calculate amplitude of the particle momentum deviation (from its actual momentum and longitudinal co-ordinate, formula (3.14)). Thereafter the phase of synchrotron oscillations is generated uniformly between 0 and  $2\pi$  and new co-ordinate and momentum has to be calculated as a function of the phase.

The bucket height can be less than the longitudinal acceptance, in this case the synchrotron motion has to be simulated for the ions outside the bucket.

## 4. Benchmarking of BETACOOOL for FAIR

For the benchmarking of the BETACOOOL code the simulation of cooling process were done for storage rings SIS18, ESR and NESR. For parameters of SIS18 the Core-Tail model of intrabeam scattering kick was used because the initial ion beam size is much larger than the electron one and during the cooling process the beam distribution has the dense core in transverse plan. ESR simulations were done for experimental parameters with internal gas cell target and new model of RF barrier bucket. Simulations for NESR with barrier bucket were compared with simulation results of the program which was elaborated by T.Katayma.

### 4.1. SIS18 simulation with large initial emittances

Parameters of simulation for SIS18 are presented in Table 4.1. Large initial beam size leads to formation of the very dense core in the transverse plan (Fig.4.1,b). For these simulations the Core-Tail model of the intrabeam scattering kick was used.

Table 4.1. SIS18 parameters.

Ion kind	$^{238}\text{U}^{73+}$
Particle number	$10^9$
Energy [MeV/u]	11.4
Circumference [m]	216.72
Initial emittance hor/ver [pi mm mrad]	50 / 20
Initial momentum spread	0.001
<b>Electron cooler</b>	
Electron cooler length [m]	3
Magnetic field [kG]	1.5
Electron beam current [A]	0.3
Electron beam radius [cm]	1.27
Transverse temperature [eV]	0.03
Longitudinal temperature [meV]	0.1
Neutralization of space charge, [%]	30

Due to the large influence of the space charge of the electron beam (Fig.4.2, c) the longitudinal profile is not symmetrical (Fig.4.2, a) and the momentum spread has some positive deviation (Fig.4.3, a).

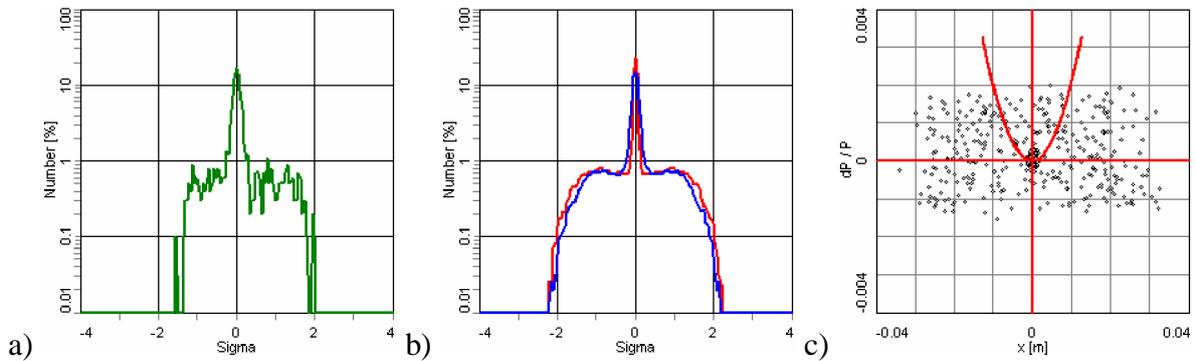


Fig.4.2. Particle distribution after 2 sec of cooling process.

a) longitudinal profile, b) transverse profiles, c) sorted invariants, c) space charge parabola of electron beam (red parabola) and particle distribution in cooler section (black dots).

Though the decreasing of the rms transverse emittances seems very slow during the cooling process (Fig.4.3, b) however more than 60% particles are cooled down after 2 seconds (Fig.3.2, a) and the dense core appears even after a few hundred milliseconds of the cooling process (Fig.4.3, c).

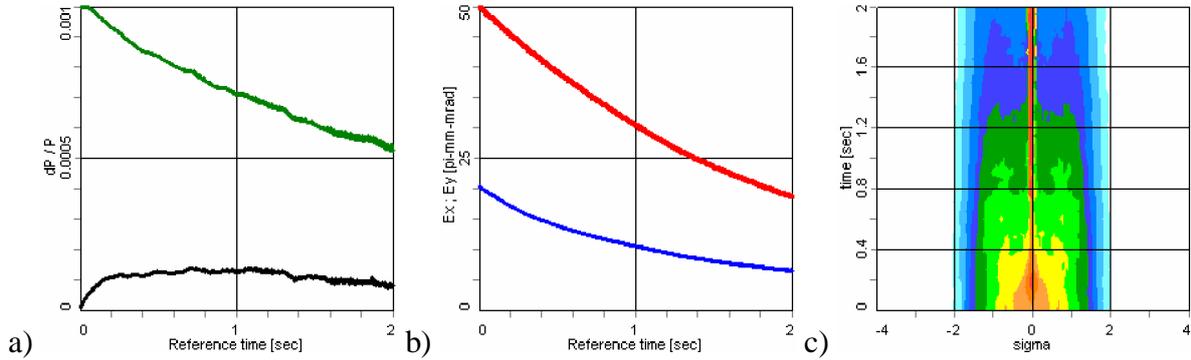


Fig.4.3. Evolution of beam parameters during cooling process.

- a) rms momentum spread (green) and momentum spread deviation (black),  
b) rms horizontal (red) and vertical (blue) emittances, c) horizontal profile.

#### 4.2. ESR simulation with barrier bucket and internal target

The cooling process with barrier bucket and internal gas sell target was simulated for ESR parameters (Table 4.2). The barrier was chosen in 3 times larger than initial rms momentum spread. Otherwise the particle losses start without any heating effects because 99% particles exist in 3 sigma of rms momentum spread. Initial transverse emittances and momentum spread were chosen near the equilibrium between the electron cooling and the heating from the intrabeam scattering and the internal target. Parameters of the internal gas cell target are presented in Fig.4.4.

Table 4.2. ESR parameters with burrier bucket

Ion kind	$^{208}\text{Pb}^{82+}$
Particle number	$1.8 \times 10^8$
Energy [MeV/u]	400
Circumference [m]	108.36
Initial emittance hor/ver [ $\pi$ mm mrad]	0.3/0.15
Initial momentum spread	$1.5 \times 10^{-4}$
Barrier height [dP/P]	$4.4 \times 10^{-4}$
RF voltage [V]	150
RF duration, $T_{\text{RF}}/T_{\text{REV}}$	0.4
Gap duration, $T_{\text{GAP}}/T_{\text{REV}}$	0.2
Bucket length [m]	38
Synchrotron period [sec]	0.007
Electron cooler length [m]	1.8
Electron beam current [A]	0.2
Electron beam radius [cm]	2.5

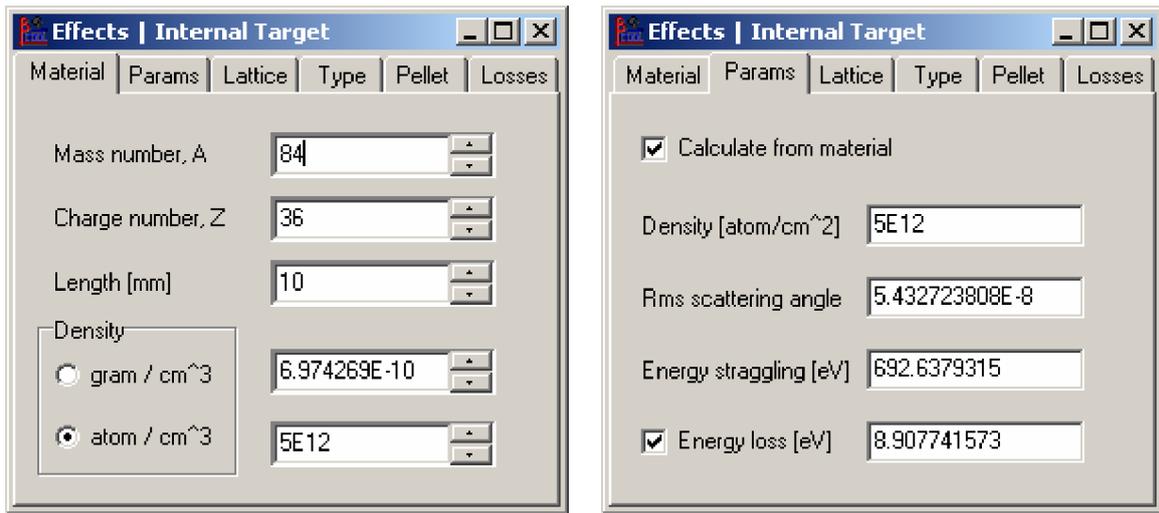


Fig.4.4. Parameters of gas cell target.

Simulations show that transverse emittances and momentum spread practically don't change during long time (Fig.4.5, a, b). Due to energy losses on the internal target the particles can pass the barrier bucket and form the long tail in the longitudinal plan (Fig.4.5, c and Fig.4.6, b) that leads to negative deviation of the momentum spread (Fig.4.5, b). In the present version of the Betacool program doesn't realize the particle dynamics over barrier bucket and particles in the tail "freeze" in the longitudinal plane (Fig.4.6, c).

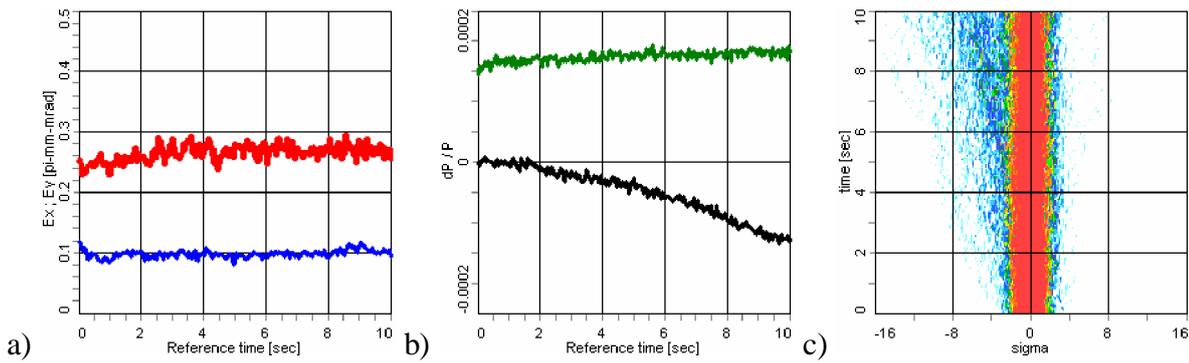


Fig.4.5. Cooling process with barrier bucket. a) emittances (red – horizontal, blue – vertical), b) momentum spread (green) and momentum deviation (black), c) momentum spread on time

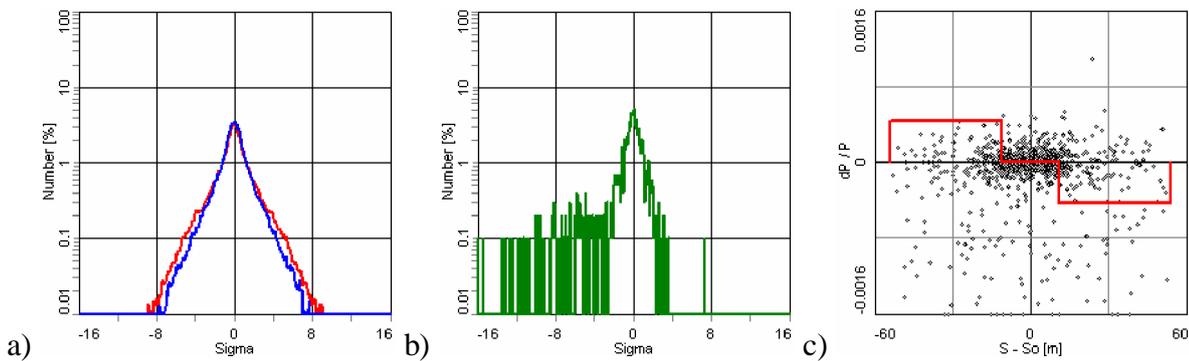


Fig.4.6. Beam distribution after 10 sec of the cooling process. a) transverse profiles, b) momentum spread distribution, c) longitudinal phase space: black dots – particles, red lines – RF bucket.

### 4.3. NESR simulation with barrier bucket

The project of NESR includes the multi stacking injection of ion beam with the moving RF barrier bucket. The present version of the BETACOOOL code has not a model of the moving barrier bucket but user can change the position and the amplitude of RF bucket manually during the simulation process. One can simulate the first cycle of injection procedure with static RF bucket. Parameters of simulation are presented in Table 1.

Table 4.3. NESR parameters

Ion kind	$^{132}\text{Sn}^{50+}$
Particle number	$10^8$
Energy [MeV/u]	740
Circumference [m]	222.11
Initial emittance [ $\pi$ mm mrad]	0.125
Initial momentum spread	$2.5 \times 10^{-4}$
Barrier height [dP/P]	$6.8 \times 10^{-4}$
RF voltage [kV]	2
RF duration, $T_{\text{RF}}/T_{\text{REV}}$	0.1
Gap duration, $T_{\text{GAP}}/T_{\text{REV}}$	0.3
Bucket length [m]	73
Synchrotron period [sec]	0.044
Electron cooler length [m]	5
Electron beam current [A]	0.1
Electron beam radius [cm]	0.75

The initial momentum spread is chosen in the correspondence with barrier height when all particle are circulated between barriers (Fig.4.7, a). Results of simulations have a good agreement with simulation results which were done with the program code written by T.Katayama (Fig.4.8) [COOL'07].

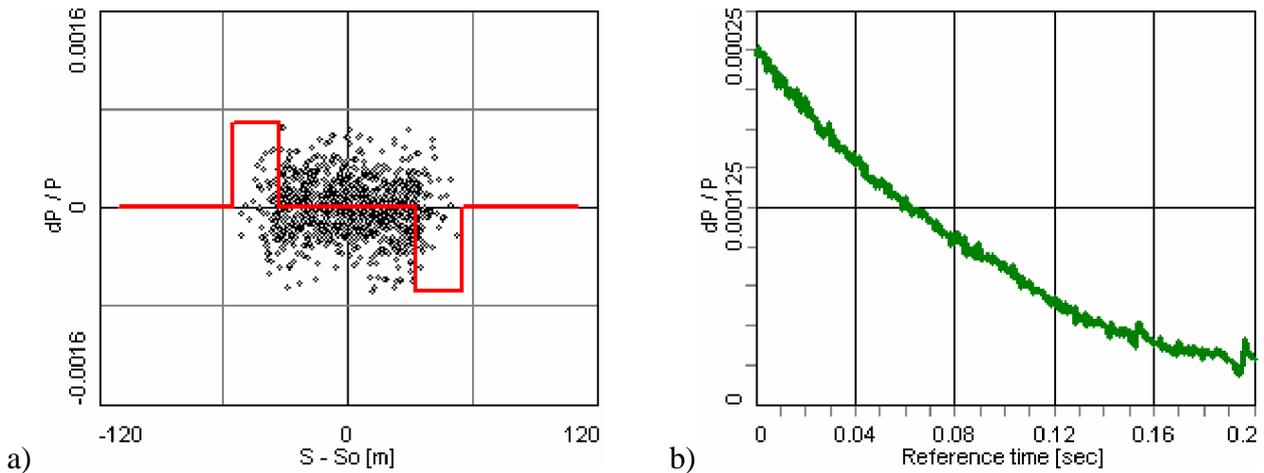
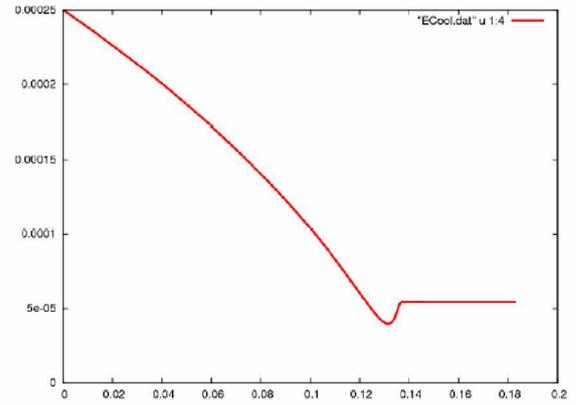
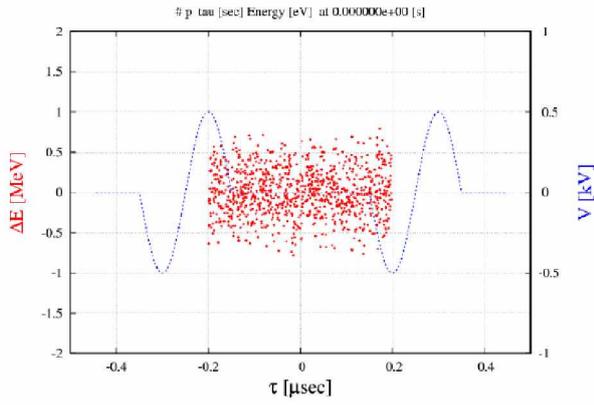


Fig.4.7. Simulation of cooling process with static barrier bucket.

- a) initial distribution of particles in longitudinal phase space, red line corresponds to width and height of bucket, b) evolution of momentum spread during cooling process.



a)

b)

Fig.4.8. Simulation by the program of T.Katayama.

a) initial distribution with sin shape of bucket, b) evolution of momentum spread.

## 5. Propositions for further development

The present version of the BETACOOOL program has the static model of the barrier bucket with rectangular shape which was resolved analytically in the longitudinal phase space. The using of this model is very difficult for the moving bucket with the arbitrary shape.

A new numerical model of the RF bucket can be implemented in the BETACOOOL code when the motion of one particle through each barrier is calculated independently. After crossing of the barrier the particle can increase energy, decrease energy or can be reflected from burrier (Fig.5.1). This model can estimate the particle motion in the case of the moving barrier also.

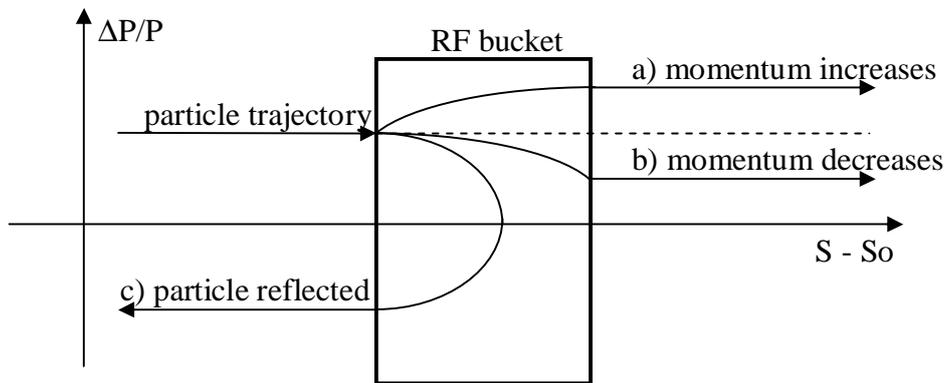


Fig.5.1. Particle trajectories trough barrier in longitudinal phase space.

For the description of the individual synchrotron motion of each particle one can use the series of the barrier buckets (Fig.5.2) and the numerical integration over longitudinal phase space. If the integration step of the Model Beam algorithm is lager than the synchrotron period, we can assume that for the static barriers the synchrotron period is constant. For the moving barriers we must calculate the particle motion during all synchrotron oscillations.

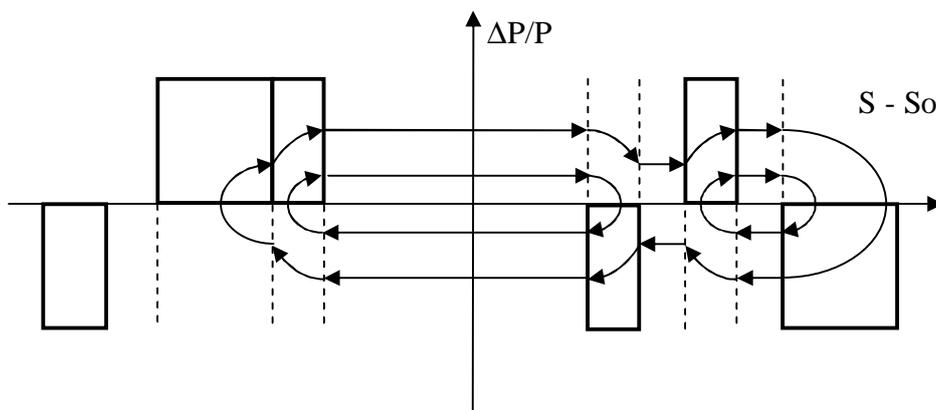


Fig.5.2. Particle trajectories in series of barrier buckets.

An approximation of the arbitrary shape of the barrier bucket can be done with the numerical integration over series of the rectangular barriers (Fig.5.3).

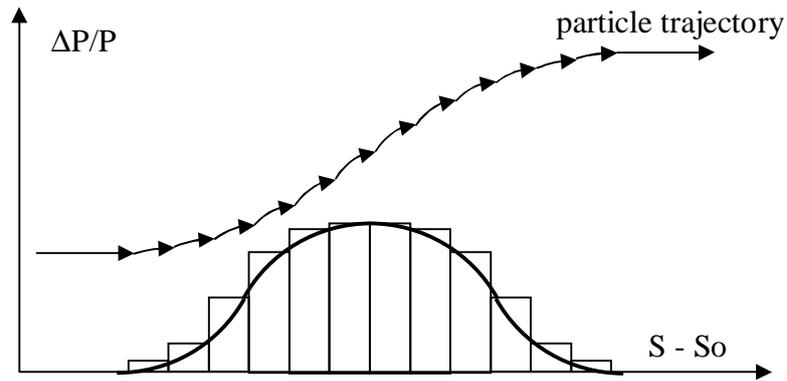


Fig.5.3. Approximation of sin shape with series of rectangular barriers.

Further improvements of BETACOOOL code are related to the update of main physical effects: model of the burrier bucket, intrabeam scattering, electron and stochastic cooling, internal target and colliding mode. Main points of further development of the BETACOOOL code are following:

- Finish the analytical model of burrier bucket (phase trajectory over bucket height)
- New model of burrier bucket with arbitrary shape and moving in time
- IBS model for arbitrary distribution of particles in longitudinal phase space
- Local model of IBS and ECOOL
- Improvement of stochastic cooling
- Description of all effects in frame of Langevin equation
- Linux version for supercomputer
- Two rings and two beams for collider mode
- Improvement of luminosity calculation for internal target and collider mode